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SINGLE-PHASE INDUCTION MOTOR TORQUE

by

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A THESIS

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The undersigned certify that they have read,
and recommend to the Faculty of Graduate Studies for
acceptance, a thesis entitled Single-Phase Induction Motor
Torque submitted by Merlyn Price in partial fulfillment
of the requirements for the degree of Master of Science.

ABSTRACT

How a single-phase induction motor operates has been a topic of discussion for over sixty years. Theories have been put forward and practical engineers have found ways to calculate the motor performance quite well. Two of these theories - the cross-field theory and the revolving-field theory - are in common use today.

Each has its advantages and disadvantages, but in each of these theories the assumptions used in obtaining the circuit equations from which the equivalent circuit is derived fail to show the presence of a double frequency pulsating torque. Kimball and Alger⁽³⁾ showed the presence of torque pulsations of double line frequency of substantial magnitude. The purpose of this study is to find a theory describing the operation of the single-phase induction motor in which the magnitude of the double frequency torque is given directly.

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BRIEF HISTORY

Long ago, mathematicians discovered that the sum of two uniformly rotating vectors equal in magnitude and rotating in opposite directions was equal at all instants of time to a stationary vector which merely pulsated in magnitude. From this, Ferraris deduced that the single-phase induction motor should have the same general torque characteristics as two polyphase induction motors of equal rating, mechanically coupled and connected for opposite directions of rotation. At any speed the net shaft torque is the algebraic sum of the torques. This approach illustrated several known facts of the single-phase motor.

1. Locked rotor torque was zero.
2. The motor will start and come up to speed equally well in whichever direction it is started.
3. No-load slip is higher than that of a comparable polyphase motor.
4. Breakdown torque is materially less than that of a comparable polyphase motor.

The principal error in the Ferraris method was the assumption of a constant voltage impressed on both motors. It was found that a much closer approximation to practical tests was obtained by treating the system as two machines connected electrically in series rather than in parallel.

One of the principal theories in use today is the revolving-field theory. This is based on the idea that the air

gap flux in the single-phase motor may be resolved into two rotating fields of an equal magnitude, revolving in opposite directions. The equations and diagrams are built up assuming that the actual stator and rotor of the single-phase motor are replaced by two stators and rotors, each equal to the original machine with the stators in series and the rotors on the same shaft.

The other well-used theory, the cross-field theory, was developed on the following lines. When rotor bars cut the field due to the stator, voltages are induced by flux cutting and the resulting currents set up a field which is essentially in space and time quadrature with the stator field. The resultant of these two fields is a true rotating field such as that obtained in a two-phase motor. The strength of the cross-field is proportional to the speed of rotation and so the locus of the field vector is an ellipse. At standstill the cross-field is zero and the locus vector is simply a vertical line. At synchronous speed the cross-field is approximately equal to the main field and the locus vector is a circle. It must be stressed that the resultant field is constant and revolves at constant speed at synchronous speed only. At all other speeds the resultant field varies in magnitude as it rotates.

It is interesting to note that the most significant points relating to the acceptance of the revolving-field theory are recorded in an A.I.E.E. paper of 1898. In his discussion on the Steinmetz paper of 1898⁽¹⁾, A.E. Kennelly suggested the revolving-field theory.

INTRODUCTION

Differences between the Single-Phase and Polyphase Motor.

(a) Locked-rotor Torque

A single-phase motor has no locked-rotor torque whereas a polyphase motor has a locked-rotor torque. In the polyphase motor locked rotor torque exists because the primary field is effectively a revolving field which induces a voltage in the rotor and the resulting currents have components which react with the primary field to produce a torque in the direction of the revolving field. In the single-phase induction motor the rotor current is produced by the transformer action of a stator flux. The flux due to the rotor current is in space phase with the air gap flux due to the stator. Hence, the torque is zero.

Also, in the polyphase motor the locked-rotor torque can be increased by increasing the rotor resistance up to the point where the locked-rotor torque is equal to the breakdown torque. In the single-phase motor no amount of rotor resistance will produce any locked-rotor torque for the very reason given above.

(b) No-load Conditions

In the polyphase motor, the no-load rotor current and rotor copper loss, I^2R , are very small whereas in the single-phase motor the quadrature field is set up by currents flowing in the rotor. Hence, there are appreciable rotor currents and rotor copper loss at no load. Secondly, if just enough

power were applied to the stator of a polyphase motor to supply friction and windage losses and small stator losses, the motor would operate at synchronous speed. However, during operation of a single-phase motor there are losses in the cross-field and so under the conditions just specified the single-phase motor would not attain synchronous speed. Hence, the no-load slip of a single-phase motor will be higher than that of a polyphase motor.

(c) Effect of Rotor Resistance on Breakdown Torque

In a polyphase motor, breakdown torque is independent of rotor resistance, whereas in a single-phase machine, the cross-field is supplied by rotor currents, the strength of which depends upon the rotor resistance. Hence, increasing the resistance of a single-phase motor decreases the cross-field flux and it will be shown that this causes a decrease in the breakdown torque.

(d) Torque Pulsations

There are no double frequency torque pulsations in a balanced polyphase motor but there are, without doubt, double frequency torque pulsations in the single-phase motor.

The theory presented in this thesis will not be based on that of the polyphase motor. Its development should lead to a direct answer for the torque of the single-phase induction motor.

The following symbols will apply:

V	=	Applied voltage.
I_1	=	Line current.
I_o	=	Primary exciting current, rotor open-circuited.
I_{2t}	=	Rotor current in the transformer axis.
I_{2s}	=	Rotor current in the speed axis.
r_1	=	Resistance of the stator winding.
r_2	=	Resistance of each of the rotor circuits.
X_o	=	Mutual magnetizing reactance of the stator and rotor windings.
x_1	=	Leakage reactance of the stator winding.
x_2	=	Leakage reactance of each of the rotor windings.
N	=	Effective number of turns in each of the circuits.
f	=	Frequency of applied voltage.
σ	=	Slip.
S	=	Speed as a fraction of synchronous speed.
ϕ_t	=	The transformer flux that is mutual to the stator winding and the rotor circuit in the transformer axis.
ϕ_{2t}	=	The leakage flux of the rotor circuit in the transformer axis.
ϕ_{2s}	=	The leakage flux of the rotor circuit in the speed axis.
ϕ_s	=	The speed flux that is mutual to the stator winding and the rotor circuit in the speed axis.
E_{tt}	=	e.m.f. induced by transformer effect of ϕ_t .
E_{tr}	=	e.m.f. generated by rotation through ϕ_s .
E'_{tr}	=	e.m.f. generated by rotation through ϕ_{2s} .

- E_{sr} = e.m.f. generated by rotation through ϕ_t .
- E'_{sr} = e.m.f. generated by rotation through ϕ_{2t} .
- E_{st} = e.m.f. induced by transformer effect of ϕ_s .
- Z_1 = $r_1 + jx_1$ = primary leakage impedance.
- Z_2 = $r_2 + jx_2$ = secondary leakage impedance of each rotor circuit.
- Z_o = $1/Y_o$ = secondary exciting impedance.
- c = Proportionality constant depending upon the number and distribution of the rotor currents and upon the frequency of the applied voltage.
- I_{2f} = Rotor current in the "forward" motor.
- I_{2b} = Rotor current in the "backward" motor.
- E_f = Voltage across the "forward" motor.
- E_b = Voltage across the "backward" motor.
- τ_f = Forward torque.
- τ_b = Backward torque.
- τ = Useful torque.
- k = Proportionality constant which makes the units of τ , ft.-lbs.

THE CROSS-FIELD THEORY AND THE REVOLVING-FIELD THEORY

The Cross-Field Theory

A brief outline of the cross-field theory is given based on the usual simplifying assumptions of sine wave distribution of m.m.f.'s; uniform air gaps; no saturation effects; no angle of lag between magnetizing current and flux.

According to the cross-field theory, the components of the main flux of the motor and the rotor currents are considered separately in two axes at right angles to each other. The axis of the stator winding is called the transformer axis and the axis at right angles to it the speed axis. A squirrel-cage is considered as equivalent to a commutated winding with brushes bearing on the commutator short-circuited on themselves in the transformer and speed axis. The motor can then be represented diagrammatically as in Figure 1.

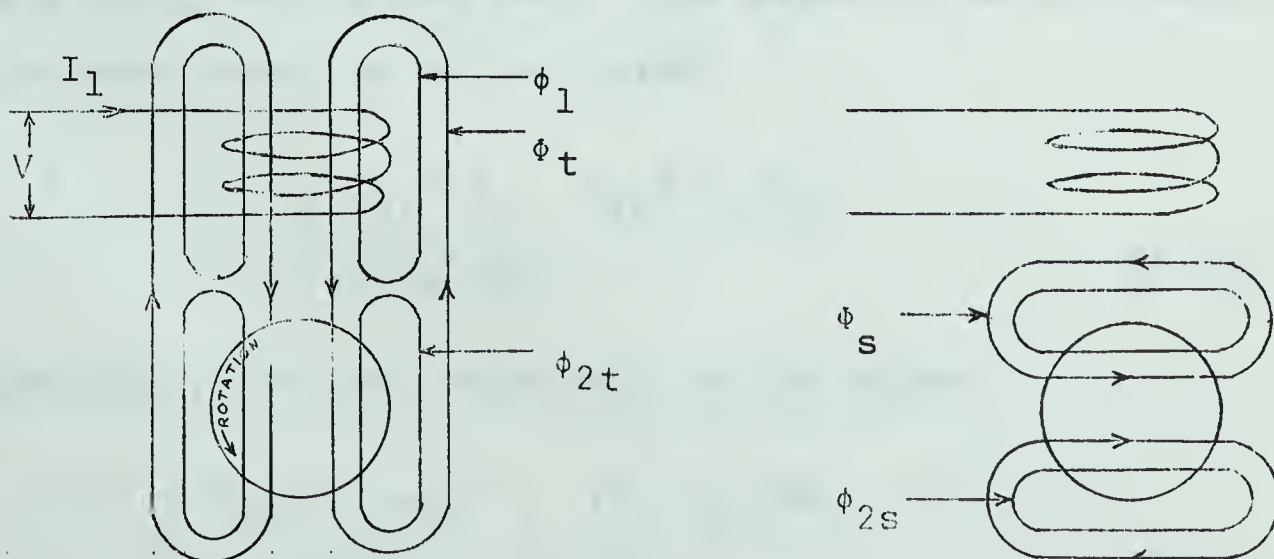


Figure 1. Flux components of a single-phase induction motor

In terms of the currents and the motor reactances, the equations for flux components are:

$$\phi_t = \frac{X_o (I_1 - I_{2t})}{2\pi f N} \quad \text{Eq. (1)}$$

$$\phi_s + \phi_{2s} = \frac{(X_o + x_2) I_{2s}}{2\pi f N} \quad \text{Eq. (2)}$$

$$\phi_1 = \frac{x_1 I_1}{2\pi f N} \quad \text{Eq. (3)}$$

$$\phi_{2t} = \frac{x_2 I_{2t}}{2\pi f N} \quad \text{Eq. (4)}$$

Voltage applied to the stator terminals must overcome the resistance drop and the mutual and leakage reactance drops due to alternation of ϕ_t and ϕ_1 . The equation is:

$$V = I_1(r_1 + jx_1) + (I_1 - I_{2t})jX_o \quad \text{Eq. (5)}$$

In the transformer axis of the rotor, the sum of the voltages induced by transformer action of ϕ_t and ϕ_{2t} , and by rotation through the flux ($\phi_s + \phi_{2s}$) plus the resistance drop $r_2 I_{2t}$ must equal zero. The equation is for counter-clockwise rotation of the rotor.

$$0 = - (I_1 - I_{2t})jX_o - I_{2s}S(X_o + x_2) + I_{2t}(r_2 + jx_2) \quad \text{Eq. (6)}$$

Similarly, for the field axis of the rotor:

$$0 = I_{2s}j(X_o + x_2) - (I_1 - I_{2t})SX_o + Sx_2 I_{2t} + r_2 I_{2s} \quad \text{Eq. (7)}$$

Solving these three voltage equations for I_1 , I_{2t} and I_{2s} (see Appendix 1) we get:

$$I_1 = \frac{E \left[-r_2^2 + (1-s^2)(X_o + x_2)^2 - j2r_2(X_o + x_2) \right]}{U_1 + jW_1} \quad \text{Eq. (8)}$$

$$I_{2t} = \frac{E X_o \left[(1-s^2)(X_o + x_2) - jr_2 \right]}{U_1 + jW_1} \quad \text{Eq. (9)}$$

$$I_{2s} = - \frac{SEX_o r_2}{U_1 + jW_1} \quad \text{Eq. (10)}$$

where

$$U_1 = -r_1 r_2^2 + 2r_2 x_1 (X_o + x_2) + r_2 X_o (X_o + 2x_2) + (1-s^2)r_1 (X_o + x_2)^2 \quad \text{Eq. (11)}$$

$$W_1 = -r_2^2 x_1 - 2r_1 r_2 (X_o + x_2) - r_2^2 X_o + (1-s^2) \left[x_1 (X_o + x_2)^2 + x_2 X_o (X_o + x_2) \right] \quad \text{Eq. (12)}$$

Substituting the above values of I_1 , I_{2t} and I_{2s} in equations (1), (2) and (4), we get:

$$\phi_t - \phi_{2t} = - \frac{EX_o \left[r_2^2 + jr_2(X_o + x_2) \right]}{2\pi fN (U_1 + jW_1)} \quad \text{Eq. (13)}$$

and

$$\phi_s + \phi_{2s} = - \frac{SEX_o (X_o + x_2)r_2}{2\pi fN (U_1 + jW_1)} \quad \text{Eq. (14)}$$

The torque developed by the motor consists of two components, one component T_f due to the interaction of the current I_{2t} and the flux $(\phi_s + \phi_{2s})$, and another component T_b due to the interaction of I_{2s} and the flux $(\phi_t - \phi_{2t})$.

From Equations (9) and (14), we get:

$$\tau_f = \frac{E^2 X_o^2 (X_o + x_2)^2 r_2 S (1 - S^2)}{U_1^2 + W_1^2} \quad \text{Eq. (15)}$$

and from Equations (10) and (13), we get:

$$\tau_b = \frac{E^2 X_o^2 r_2^3 S}{U_1^2 + W_1^2} \quad \text{Eq. (16)}$$

and the total torque developed by the motor is:

$$\tau = \frac{E^2 X_o^2 r_2 S \left[(1 - S^2) (X_o + x_2)^2 - r_2^2 \right]}{U_1^2 + W_1^2} \quad \text{Eq. (17)}$$

Thus with the assumptions used, the expression for the net torque derived from the basic equations fails to show the presence of any double frequency components.

Revolving-Field Theory

A similar outline to that given for the cross-field theory will now be given for the revolving-field theory.

The actual pulsating air-gap flux is resolved into two equal counter-revolving fields. Half the voltage induced by the air-gap flux is due to the forward field and half to the backward field. Half the mutual reactance is charged to the forward and half to the backward field. Rotor resistance and rotor leakage reactance are also divided into two equal parts.

To derive the equivalent circuit, consider the rotor revolving at a speed giving a slip of σ with respect to the forward field.

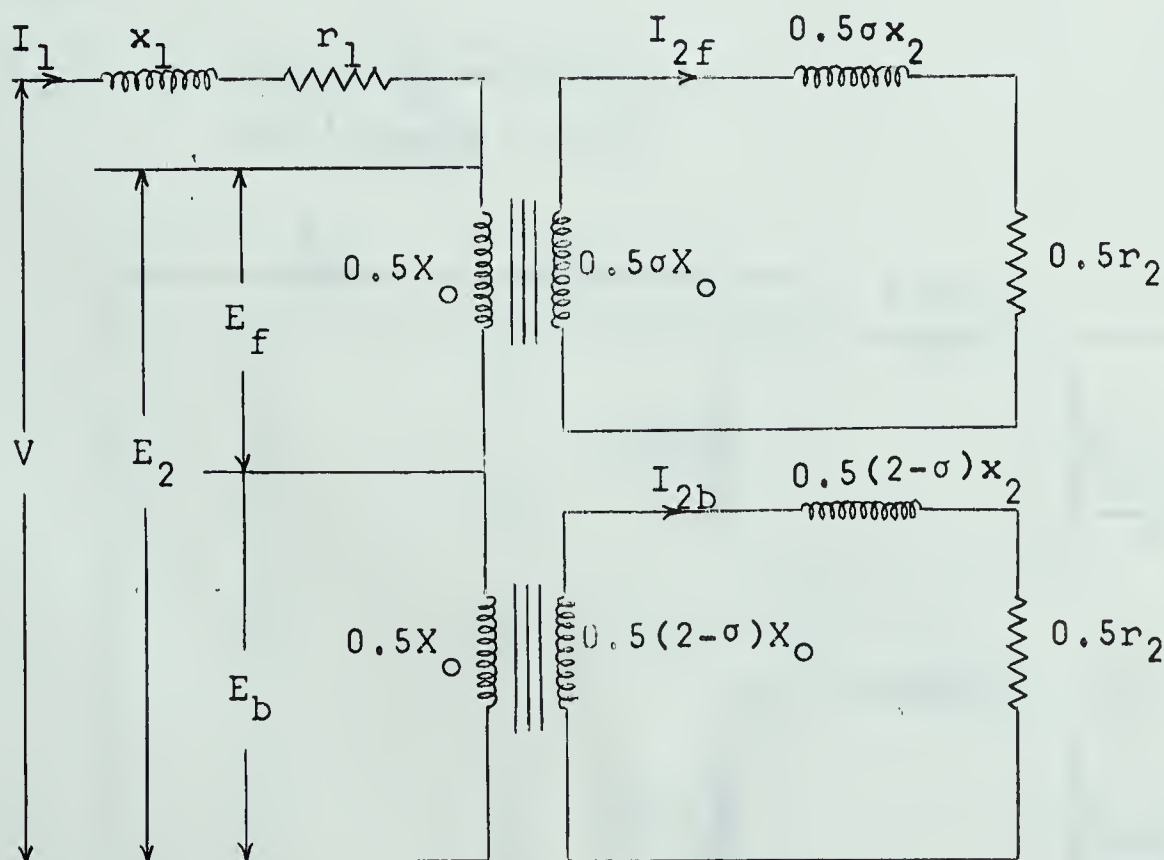


Figure 2. Coupled circuits representing the Single-Phase Induction Motor

The circuit equations (see Fig.2) are as follows:

For the primary:

$$\begin{aligned} V &= I_1(r_1 + jx_1) + j(I_1 - I_{2f})(0.5X_o) + j(I_1 - I_{2b})(0.5X_o) \\ &= I_1(r_1 + jx_1) + E_f + E_b = I_1z_1 + E_2 \quad \text{Eq.(18)} \end{aligned}$$

For the forward-field secondary circuit:

$$0 = -\sigma E_f + I_{2f}(r_2 + j\sigma x_2)(0.5)$$

i.e.

$$I_{2f} = \frac{E_f}{(0.5)\left(\frac{r_2}{\sigma} + jx_2\right)} \quad \text{Eq.(19)}$$

For the backward-field secondary circuit:

$$0 = -(2-\sigma)E_b + I_{2b}[r_2 + j(2-\sigma)x_2](0.5)$$

i.e.

$$I_{2b} = \frac{E_b}{(0.5)\left[\frac{r_2}{(2-\sigma)} + jx_2\right]} \quad \text{Eq.(20)}$$

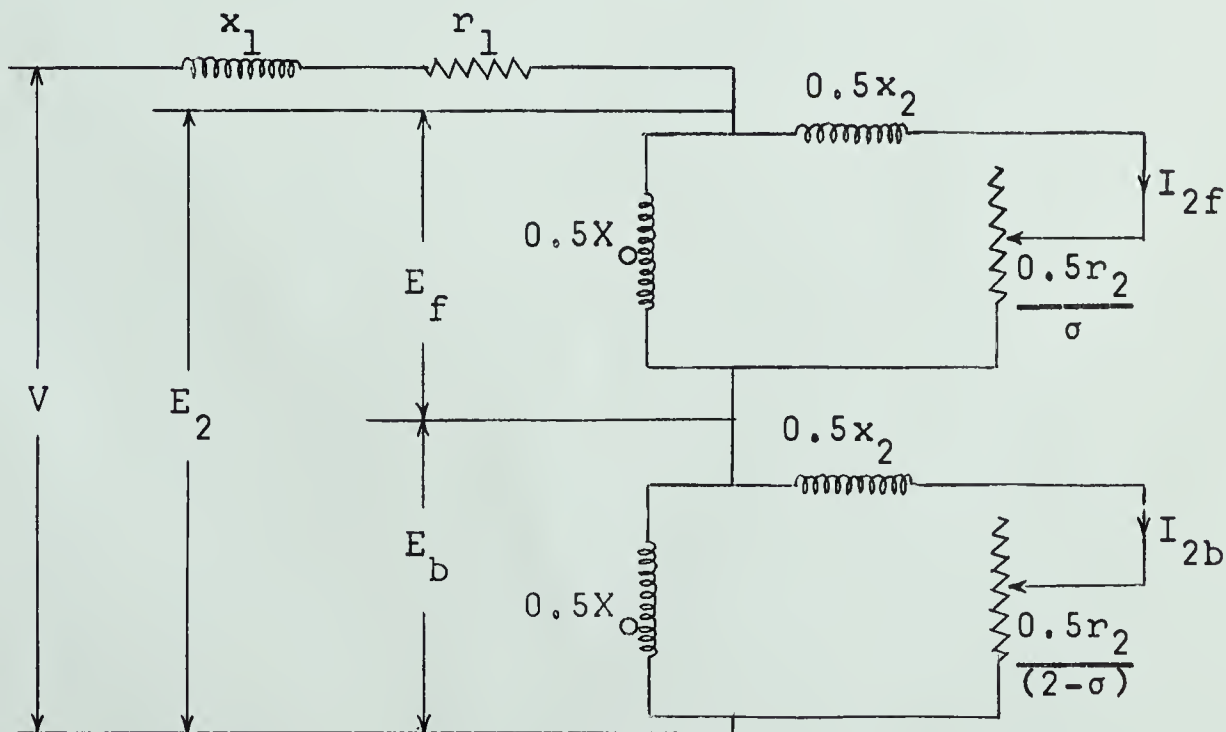


Figure 3. Conventional Equivalent Circuit for the Single-Phase Induction Motor, according to the Revolving-Field Theory

The forward torque in synchronous watts, following the reasoning which results from polyphase induction motor analyses is:

$$\tau_f = I_{2f}^2 \frac{(0.5)r_2}{\sigma} \quad (\text{synchronous watts}) \quad \text{Eq. (21)}$$

$$\tau_b = I_{2b}^2 \frac{(0.5)r_2}{2-\sigma} \quad (\text{synchronous watts}) \quad \text{Eq. (22)}$$

$$\tau = \tau_f - \tau_b = I_{2f}^2 \frac{(0.5)r_2}{\sigma} - I_{2b}^2 \frac{(0.5)r_2}{2-\sigma} \quad \text{Eq. (23)}$$

Once more, with the assumptions used, it is seen that the expression for the net torque derived from the basic equations fails to show the presence of any double frequency components.

NEW THEORY

In the development of this theory, the angle of hysteretic lag between flux and m.m.f. is neglected. Corrections for core loss can be made by treating the core loss as though it were a friction loss. Should it be desirable, the angle of hysteretic lag can be taken into account by using the complex quantity $Z_o = R_o + jX_o$ for the magnetizing impedance in place of the pure reactance jX_o , which is used in the equations. The inclusion of R_o would complicate matters slightly by adding to the lengths of the equations, and would yield but a very slight increase in accuracy. Sine wave distribution of m.m.f. and uniform air-gap permeance are assumed in all cases. A further assumption is that there are no saturation effects.

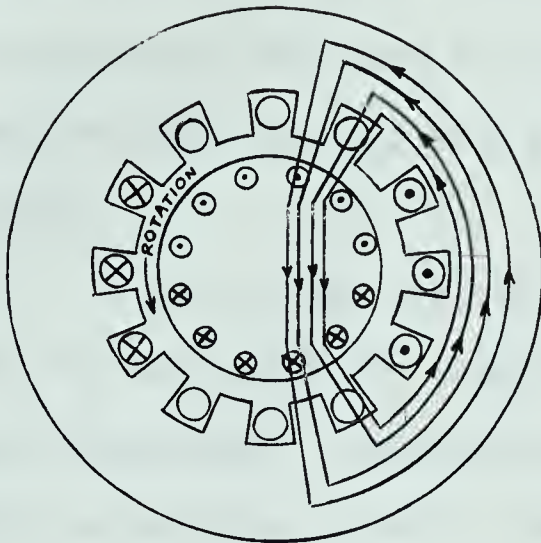


Figure 4 (a)

Stator field and
induced rotor currents

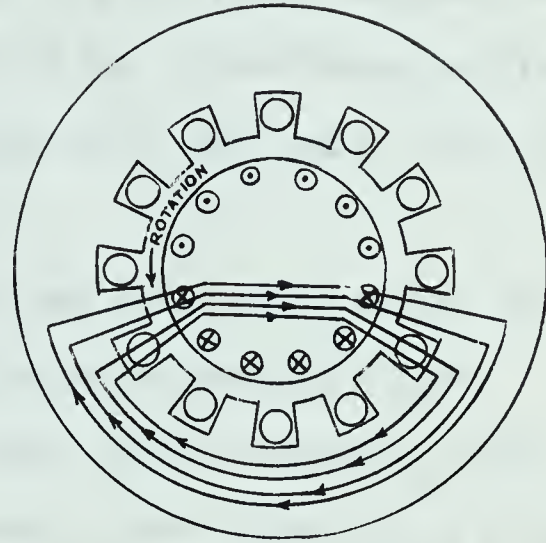


Figure 4 (b)

Magnetic field due to
rotor currents

Figure 4 (a) shows the exciting current and the field produced by it at a given instant in time. This field varies sinusoidally in time with the exciting current but the axis of the field will not change.

The rotor winding can be thought of as two single coils at right angles to each other, as shown in Figure 5.

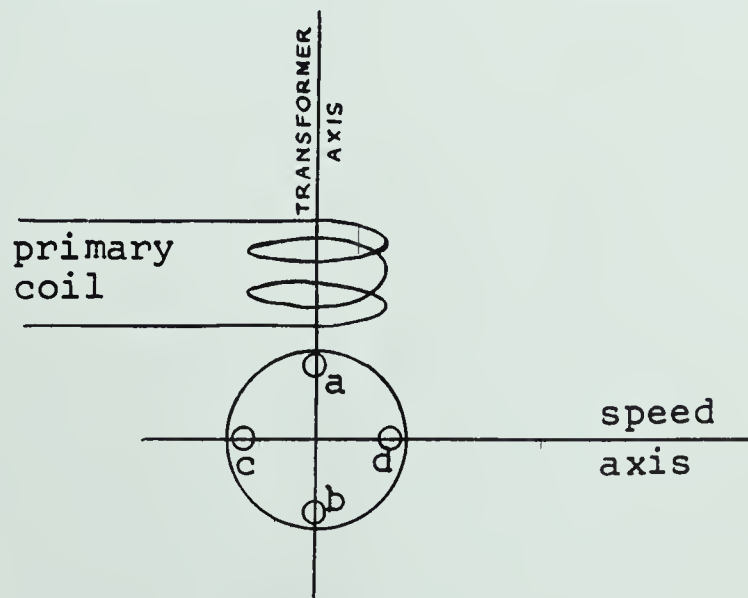


Figure 5

At standstill the axis of coil 'ab' is at right angles to the transformer axis and so there is no transformer action between the flux due to the primary coil and coil 'ab' on the rotor.

Let it be assumed that an external torque is applied to the rotor so that the rotor is made to turn counter-clockwise. Rotation of coil 'ab' through ϕ_t will generate a voltage in coil 'ab' and, since there is a closed path, a current will flow in the direction shown in Fig.4(b). The current in 'ab' sets up a flux ϕ_s in the speed axis. Coil 'cd' has an e.m.f. induced in it by virtue of its motion through the flux ϕ_s and, additionally, its axis is in line with the transformer axis so it acts like a single turn secondary coil of a transformer. Hence, there is an e.m.f. in coil 'cd', due to transformer action, in opposition to that in the primary coil.

Phase Relationships

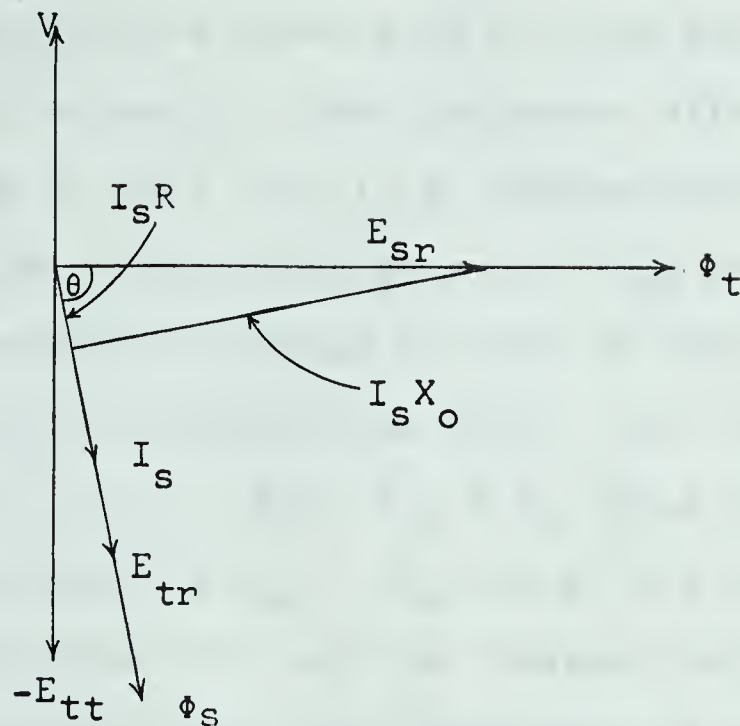


Figure 6. Phasor Time Diagram

Consider now the phase relations between the various fluxes, e.m.f.'s and currents starting with flux ϕ_t in any arbitrarily selected position, say, horizontal. Alternation of this flux induces an e.m.f. E_{tt} and, in accordance with basic transformer theory, the phase of E_{tt} lags 90° behind that of ϕ_t . At the same instant, rotation of coil 'ab' in a counter-clockwise direction through ϕ_t generates an e.m.f. E_{sr} in time phase with ϕ_t . Since there is no winding on the stator or any other part of the motor on which the current due to E_{sr} can act, the rotor behaves, as far as this current and the speed axis are concerned, like a reactor. It is equivalent to a single winding on a magnetic circuit with two air gaps. Therefore, its reactance is high and the current, I_s , due to E_{sr} lags E_{sr} by approximately 90° . This current I_s

sets up a flux ϕ_s in the speed axis in time phase with it. Since I_s is almost in time quadrature with ϕ_t so too is ϕ_s . Rotation of coil 'cd' in a counter-clockwise direction through ϕ_s generates an e.m.f. E_{tr} in time phase with ϕ_s . The resultant voltage in 'cd' is that due to E_{tr} and E_{tt} and its maximum value is E_t . Let the rotor current due to E_t be I_t . Now, $E_{sr} = E_s$ since there is no voltage corresponding to E_{st} . E_s and E_t are the resultant voltages in the coils 'ab' and 'cd' respectively. Neglecting leakage reactance, the rotor impedance is $R+jX_o$. The angle θ defined by $\tan\theta = \frac{X_o}{R}$ is the angle by which I_s lags E_s . I_t also lags E_t by the same angle θ .

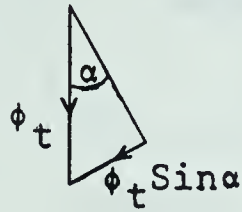
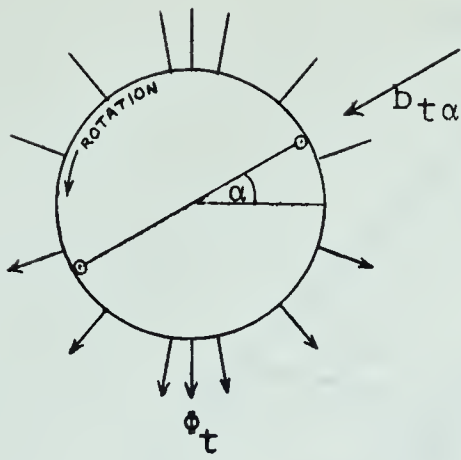


Figure 7 (a)

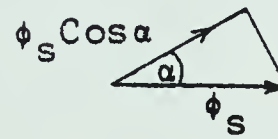
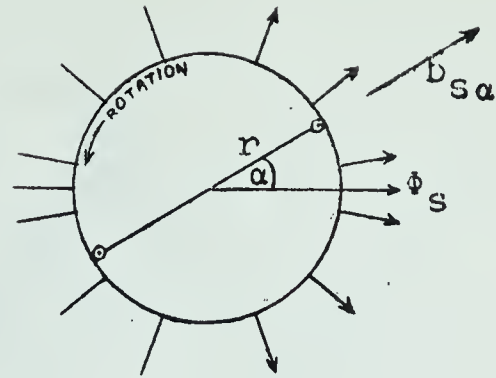


Figure 7 (b)

Flux density is a function of time and space.

$$b_{t\alpha} = B_t \sin \omega_s t \sin \alpha \quad \text{Eq. (24)}$$

$$b_{s\alpha} = B_s \sin(\omega_s t - \theta) \cos \alpha \quad \text{Eq. (25)}$$

$$E_{sr} = c \phi_t \omega_r \quad \text{Eq. (26)}$$

$$\text{Now } I_{s0} X_o = E_{sr} \sin \theta$$

$$\text{i.e. } I_{ss0} \omega_s L = E_{sr} \sin \theta$$

$$\omega_s \phi_s = E_{sr} \sin \theta$$

$$= c \omega_r \phi_t \sin \theta$$

$$\therefore \phi_s = \frac{\omega_r}{\omega_s} \phi_t \sin \theta \quad \text{Eq. (27)}$$

$$\phi_t = \phi_t \sin \omega_s t$$

$$E_{tt} = c \text{ max. value of } \frac{d\phi_t}{dt} = k \omega_s \phi_t \quad \text{Eq. (28)}$$

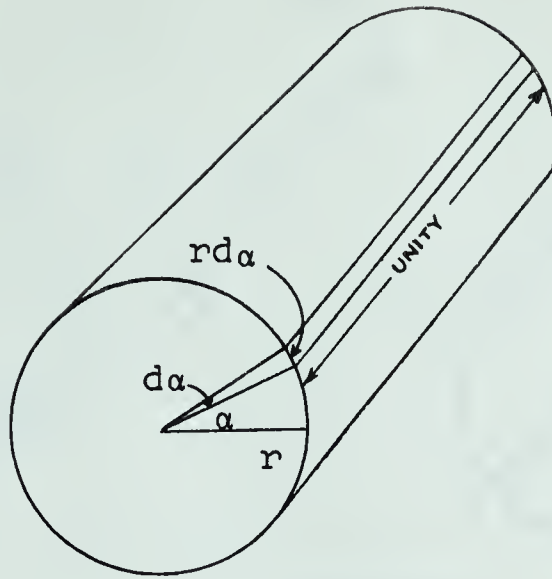


Figure 8

Consider a rotor of unit length.

$$\begin{aligned}
 \phi_t &= \phi_t \sin \omega_s t \\
 &= \int_0^\pi b_{ta} r d\alpha \\
 &= r B_t \sin \omega_s t [\cos \alpha]_0^\pi \\
 &= 2r B_t \sin \omega_s t \\
 \therefore B_t &= \frac{\phi_t}{2r} \quad \text{Eq. (29)}
 \end{aligned}$$

Also

$$\begin{aligned}
 \phi_s &= \phi_s \sin(\omega_s t - \theta) \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b_{sa} r d\alpha \\
 &= r B_s \sin(\omega_s t - \theta) [\sin \alpha]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= 2r B_s \sin(\omega_s t - \theta) \\
 \therefore B_s &= \frac{\phi_s}{2r} \quad \text{Eq. (30)}
 \end{aligned}$$

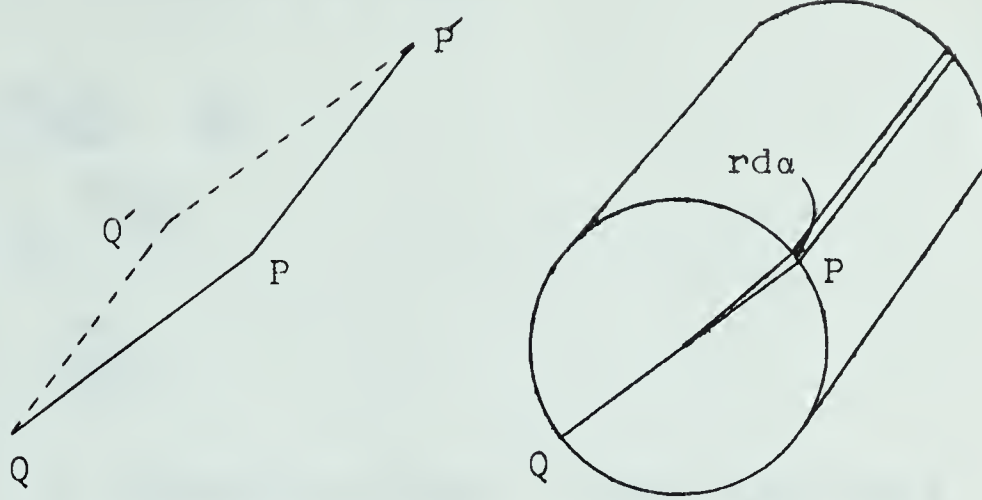


Figure 9

Let the total flux in the loop $P'PQQ'$ at a given instant in time, t say, be $\phi_{i\alpha}$ and let the corresponding instantaneous voltage be $e_{i\alpha}$. Then:

$$\phi_{i\alpha} = \int_{\alpha}^{\pi+\alpha} b_{t\alpha} r d\alpha + \int_{\pi+\alpha}^{\alpha} b_{s\alpha} r d\alpha$$

$$= rB_t \sin \omega_s t [\cos \alpha]_{\pi+\alpha}^{\alpha} + rB_s \sin(\omega_s t - \theta) [\sin \alpha]_{\pi+\alpha}^{\alpha}$$

$$= 2rB_t \sin \omega_s t \cos \alpha + 2rB_s \sin(\omega_s t - \theta) \sin \alpha$$

$$\therefore \phi_{i\alpha} = \phi_t \sin \omega_s t \cos \alpha + \phi_s \sin(\omega_s t - \theta) \sin \alpha \quad \text{Eq. (31)}$$

Since α is a function of time

$$\begin{aligned}\frac{df(\alpha)}{dt} &= \frac{df(\alpha)}{d\alpha} \cdot \frac{d\alpha}{dt} \\ &= \omega_r \frac{df(\alpha)}{d\alpha}\end{aligned}$$

$$\begin{aligned}\therefore e_{i\alpha} &= c \frac{d\phi_{i\alpha}}{dt} \\ &= c \left[\begin{aligned} &\phi_t \sin \omega_s t (-\omega_r \sin \alpha) + \cos \alpha \omega_s \phi_t \cos \omega_s t \\ &+ \phi_s \sin(\omega_s t - \theta) (\omega_r \cos \alpha) + \sin \alpha \omega_s \phi_s \cos(\omega_s t - \theta) \end{aligned} \right] \text{Eq. (32)}\end{aligned}$$

Let ρ be the surface resistivity of the copper of the rotor. Then, neglecting end-turn resistance, the resistance of the loop PP'QQ' is given by $\frac{2\rho l}{rd\alpha}$.

Hence, if $i_{i\alpha}$ is the current due to $e_{i\alpha}$:

$$i_{i\alpha} = \frac{e_{i\alpha}}{\frac{2\rho l}{rd\alpha}} = \frac{e_{i\alpha} rd\alpha}{2\rho} \quad \text{Eq. (33)}$$

Initially, the direction of the fluxes ϕ_t and ϕ_s were such that $b_{t\alpha}$ and $b_{s\alpha}$ were in opposite directions at the point P (see Fig.7) so that the instantaneous flux density, $b_{i\alpha}$ say, along the rotor at PP' is given by $b_{t\alpha} - b_{s\alpha}$.

$$\therefore b_{i\alpha} = B_t \sin \omega_s t \sin \alpha - B_s \sin(\omega_s t - \theta) \cos \alpha$$

$$\text{Torque} = \text{Force} \times \text{radius} = (Bil)r$$

If $\tau_{i\alpha}$ is the corresponding instantaneous value of torque acting on the loop PP'QQ'

$$\begin{aligned}\tau_{i\alpha} &= b_{i\alpha} i_{i\alpha} l r k \\ &= \frac{k \left[B_t \sin \omega_s t \sin \alpha - B_s \sin(\omega_s t - \theta) \cos \alpha \right] e_{i\alpha} r^2 d\alpha}{2\rho} \\ &= \frac{k r^2 d\alpha}{2\rho} \left[\left\{ B_t \sin \omega_s t \sin \alpha - B_s \sin(\omega_s t - \theta) \cos \alpha \right\} \right. \\ &\quad \left. \left\{ -\omega_r \phi_t \sin \omega_s t \sin \alpha + \omega_s \phi_t \cos \omega_s t \cos \alpha \right\} \right. \\ &\quad \left. + \left\{ B_t \sin \omega_s t \sin \alpha - B_s \sin(\omega_s t - \theta) \cos \alpha \right\} \right. \\ &\quad \left. \left\{ \omega_r \phi_s \sin(\omega_s t - \theta) \cos \alpha + \omega_s \phi_s \cos(\omega_s t - \theta) \sin \alpha \right\} \right]\end{aligned}$$

If τ_{im} is the instantaneous value of torque for the machine:

$$\begin{aligned}\tau_{im} &= 2 \int_0^\pi \tau_{i\alpha} d\alpha \\ \text{Now, } \int_0^\pi \sin \alpha \cos \alpha d\alpha &= 0 \\ \text{and } \int_0^\pi \sin^2 \alpha d\alpha &= \int_0^\pi \cos^2 \alpha d\alpha = \frac{\pi}{2}\end{aligned}$$

$$\tau_{im} = \frac{k\pi r^2}{2\rho} \left[-\omega_r B_t \phi_t \sin^2 \omega_s t - \omega_s B_s \phi_t \sin(\omega_s t - \theta) \cos \omega_s t \right. \\ \left. - \omega_r B_s \phi_s \sin^2(\omega_s t - \theta) + \omega_s B_t \phi_s \sin \omega_s t \cos(\omega_s t - \theta) \right]$$

Substituting

$$B_t = \frac{\phi_t}{2r} \quad \text{Eq. (29)}$$

$$\text{and } B_s = \frac{\phi_s}{2r} \quad \text{Eq. (30)}$$

$$\tau_{im} = \frac{k\pi r}{2\rho} \left[-\omega_r \phi_t^2 \sin^2 \omega_s t - \omega_s \phi_s \phi_t \sin(\omega_s t - \theta) \cos \omega_s t \right. \\ \left. - \omega_r \phi_s^2 \sin^2(\omega_s t - \theta) + \omega_s \phi_s \phi_t \sin \omega_s t \cos(\omega_s t - \theta) \right]$$

$$\tau_{im} = \frac{k\pi r}{2\rho} \left[-\omega_s \phi_s \phi_t \sin \theta - \omega_r \phi_t^2 \sin^2 \omega_s t \right. \\ \left. - \omega_r \phi_s^2 \sin^2(\omega_s t - \theta) \right] \quad \text{Eq. (34)}$$

The average useful torque for the machine, τ , is given by

$$\tau = \frac{1}{T'} \int_0^{T'} \tau_{im} dt$$

where T' is the period of the torque expression.

$$\tau = \frac{k\pi r}{4\rho} \left[\omega_s \phi_s \phi_t \sin \theta - \frac{\omega_r \phi_t^2}{2} - \frac{\omega_r \phi_s^2}{2} + \frac{\omega_r \phi_t^2 \sin 2\omega_s t}{4\omega_s T'} + \frac{\omega_r \phi_s^2}{4\omega_s T'} \{ \sin 2(\omega_s t - \theta) - \sin(-2\theta) \} \right] \quad \text{Eq. (35)}$$

Substituting $\phi_s = \frac{\omega_r}{\omega_s} \phi_t \sin \theta$ Eq. (27)

$$\begin{aligned} \tau &= \frac{k\pi r}{4\rho} \left[\omega_r \phi_t^2 \sin^2 \theta - \frac{\omega_r \phi_t^2}{2} - \frac{\omega_r^3 \phi_t^2 \sin^2 \theta}{2\omega_s^2} \right] \\ &= \frac{k\pi r \phi_t^2 \omega_r}{8\rho} \left[2 \sin^2 \theta - 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \right] \\ &= \frac{k\pi r \phi_t^2 \omega_r}{8\rho} \left[\sin^2 \theta - (1 - \sin^2 \theta) - \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \right] \\ &= \frac{k\pi r \phi_t^2 \omega_r}{8\rho} \left[\left\{ 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \right\} \sin^2 \theta - \cos^2 \theta \right] \quad \text{Eq. (37)} \end{aligned}$$

Let τ_1 be the average torque over the first half period of time i.e. from 0 to $\frac{T'}{2}$.

Let τ_2 be the average torque over the second half period of time i.e. from $\frac{T'}{2}$ to T' .

$$\begin{aligned}
 \tau_1 &= \frac{2}{T'} \int_0^{\frac{T'}{2}} \frac{k\pi r}{4\rho} \left[\omega_s \phi_s \phi_t \sin \theta - \omega_r \phi_t^2 \sin^2 \omega_s t - \omega_r \phi_s^2 \sin^2 (\omega_s t - \theta) \right] dt \\
 &= \frac{k\pi r}{2\rho T'} \left[\omega_s \phi_s \phi_t \sin \theta \left(\frac{T'}{2} \right) - \omega_r \phi_t^2 \left\{ \frac{1}{2\omega_s} \left(\frac{\omega_s T'}{2} - \frac{\sin \omega_s T'}{2} \right) \right\} \right. \\
 &\quad \left. - \omega_r \phi_s^2 \left\{ \frac{1}{2\omega_s} \left(\frac{\omega_s T' - \sin(\omega_s T' - 2\theta)}{2} - \sin 2\theta \right) \right\} \right] \\
 &= \frac{k\pi r}{4\rho} \left[\omega_s \phi_s \phi_t \sin \theta - \frac{\omega_r \phi_t^2}{2} - \frac{\omega_r \phi_s^2}{2} \right. \\
 &\quad \left. + \frac{\omega_r \phi_t^2 \sin \omega_s T'}{2\omega_s T'} + \frac{\omega_r \phi_s^2}{2\omega_s T'} \left\{ \sin(\omega_s T' - 2\theta) + \sin 2\theta \right\} \right] \\
 &= \frac{k\pi r}{4\rho} \left[\omega_s \phi_s \phi_t \sin \theta - \frac{\omega_r \phi_t^2}{2} - \frac{\omega_r \phi_s^2}{2} + \frac{\omega_r \phi_s^2}{\pi} \sin 2\theta \right]
 \end{aligned}$$

Substituting $\phi_s = \frac{\omega_r}{\omega_s} \phi_t \sin \theta$

see Eq.(27)

$$\begin{aligned}
 \tau_1 &= \frac{k\pi r \omega_r \phi_t^2}{8\rho} \left[2 \sin^2 \theta - 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta + \frac{2}{\pi} \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \sin 2\theta \right] \\
 \tau_1 &= \frac{k\pi r \omega_r \phi_t^2}{8\rho} \left[\left\{ 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \right\} \sin^2 \theta - \cos^2 \theta + \frac{2}{\pi} \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \sin 2\theta \right] \text{ Eq.(39)}
 \end{aligned}$$

$$\begin{aligned}
 \tau_2 &= \frac{2}{T'} \int_{\frac{T'}{2}}^{T'} \frac{k\pi r}{4\rho} \left[\omega_s \phi_s \phi_t \sin \theta - \omega_r \phi_t^2 \sin^2 \omega_s t - \omega_r \phi_s^2 \sin^2 (\omega_s t - \theta) \right] dt \\
 &= \frac{k\pi r}{2\rho T'} \left[\omega_s \phi_s \phi_t \sin \theta \left(\frac{T'}{2} \right) \right. \\
 &\quad - \omega_r \phi_t^2 \left\{ \frac{1}{2\omega_s} \left(\omega_s T' - \frac{\sin 2\omega_s T'}{2} - \frac{\omega_s T'}{2} + \frac{\sin \omega_s T'}{2} \right) \right\} \\
 &\quad - \omega_r \phi_s^2 \left\{ \frac{1}{2\omega_s} \left(\omega_s T' - \theta - \frac{\sin(2\omega_s T' - 2\theta)}{2} \right. \right. \\
 &\quad \left. \left. - \frac{\omega_s T'}{2} + \theta + \frac{\sin(\omega_s T' - 2\theta)}{2} \right) \right\} \left. \right] \\
 &= \frac{k\pi r}{4\rho} \left[\omega_s \phi_s \phi_t \sin \theta - \frac{\omega_r \phi_t^2}{2} - \frac{\omega_r \phi_s^2}{2} + \frac{\omega_r \phi_t^2}{2\omega_s T'} (\sin 2\omega_s T' + \sin \omega_s T') \right. \\
 &\quad \left. + \frac{\omega_r \phi_s^2}{2\omega_s T'} \{ \sin(2\omega_s T' - 2\theta) - \sin(\omega_s T' - 2\theta) \} \right] \\
 &= \frac{k\pi r}{4\rho} \left[\omega_s \phi_s \phi_t \sin \theta - \frac{\omega_r \phi_t^2}{2} - \frac{\omega_r \phi_s^2}{2} - \frac{\omega_r \phi_s^2}{\pi} \sin 2\theta \right]
 \end{aligned}$$

Substituting $\phi_s = \frac{\omega_r}{\omega_s} \phi_t \sin \theta$ Eq. (27)

$$\begin{aligned}
 \tau_2 &= \frac{k\pi r \omega_r \phi_t^2}{8\rho} \left[2 \sin^2 \theta - 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta - \frac{2}{\pi} \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \sin 2\theta \right] \\
 &= \frac{k\pi r \omega_r \phi_t^2}{8\rho} \left[\left\{ 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \right\} \sin^2 \theta - \cos^2 \theta \right. \\
 &\quad \left. - \frac{2}{\pi} \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \sin 2\theta \right]
 \end{aligned}$$

Eq. (40)

$$\text{Let } \tau_p = \frac{k\pi r \omega_r \phi_t^2}{8\rho} \left[\frac{2}{\pi} \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \sin 2\theta \right] \quad \text{Eq. (41)}$$

$$\text{and } \tau = \frac{k\pi r \omega_r \phi_t^2}{8\rho} \left[\left\{ 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \right\} \sin^2 \theta - \cos^2 \theta \right]$$

$$\text{Then } \tau_1 = \tau + \tau_p$$

$$\tau_2 = \tau - \tau_p$$

Hence, τ_p is the average value of the pulsating torque and τ is the average value of the useful torque.

$$\begin{aligned} \tau_p &= \frac{k\pi r \omega_r \phi_t^2}{8\rho} \left[\frac{2}{\pi} \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta \sin 2\theta \right] \\ &= C_1 \frac{\omega_r^3}{\omega_s^2} \sin^2 \theta \sin 2\theta \text{ where } C_1 \text{ is independent of } \theta \end{aligned}$$

$$= C_1 \frac{\omega_r^3}{\omega_s^2} 2 \sin^3 \theta \cos \theta$$

$$\frac{\partial \tau_p}{\partial \theta} = 2 C_1 \left[\frac{\omega_r^3}{\omega_s^2} 3 \sin^2 \theta \cos^2 \theta - \sin^4 \theta \right]$$

$$\frac{\partial \tau_p}{\partial \theta} = 0 \text{ if } \omega_r = 0 \text{ or if } 3 \sin^2 \theta \cos^2 \theta = \sin^4 \theta$$

$$\text{i.e. if } 3 \sin^2 \theta - 3 \sin^4 \theta = \sin^4 \theta$$

$$4 \sin^4 \theta = 3 \sin^2 \theta$$

$$\sin^2 \theta = \frac{3}{4}$$

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ \quad \text{since } \theta \leq 90^\circ$$

$$\begin{aligned} \frac{\partial^2 \tau_p}{\partial \theta^2} &= 2C_1 \frac{\omega_r^3}{\omega_s^2} \left[-6 \sin^3 \theta \cos \theta + 6 \sin \theta \cos^3 \theta - 4 \sin^3 \theta \cos \theta \right] \\ &= \frac{2C_1 \omega_r^3}{\omega_s^2} 6 \sin \theta \cos \theta \left[1 - \frac{8}{3} \sin^2 \theta \right] \end{aligned}$$

which is negative for $\theta = 60^\circ$

Thus, for variations in θ only, τ_p is a maximum when $\theta = 60^\circ$.

AGREEMENT BETWEEN NEW THEORY AND KNOWN FACTS

Double Frequency Torque

It is an actual physical fact that there are double frequency currents and torques in the single-phase induction motor. Test results on the secondary circuit of a single-phase induction motor at small load, taken by an oscillograph, are shown in a paper by B. G. Lamme⁽²⁾.

Little work has been done on measuring the value of the double frequency currents and torques and little has been said on the factors affecting these quantities. Consequently, it is not possible to use the theory developed and check the calculated results against known practical results. Nevertheless, the theory developed, shows quite clearly the presence of a double frequency torque and a double frequency secondary voltage (see Appendix 2).

Examination of Equations (38), (39) and (40) shows that Equation (41) is the expression for the double frequency torque component. It can be seen that Equation (41) is zero when $\omega_r = 0$. Thus, there is no average double frequency torque at standstill, but an appreciable double frequency torque would be experienced even on no-load since ω_r is then quite large. These facts are in accordance with an investigation carried out by A.L. Kimball, Jr., and P.L. Alger⁽³⁾ on Single-Phase Motor-Torque Pulsation. In this investigation it was proved that there was a double frequency torque, which was torsional, which existed even when the motor was running light, and was electromagnetic in its origin.

Zero Torque below Synchronous Speed

In the polyphase machine the torque-speed curve cuts the speed axis at 0 r.p.m. and at synchronous speed whereas the corresponding curve for the single-phase induction motor cuts the speed axis at 0 r.p.m. and at a point slightly below synchronous speed. This, then, is another characteristic that must be borne out.

Recall,

$$\tau = \frac{k\pi r \phi_t^2 \omega_r}{8\rho} \left[\left\{ 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \right\} \sin^2 \theta - \cos^2 \theta \right] \quad \text{Eq. (37)}$$

$$\tau = 0 \quad \text{a) when } \omega_r = 0$$

$$\text{b) when } 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 = \cot^2 \theta$$

$$\text{i.e. when } \left(\frac{\omega_r}{\omega_s} \right)^2 = 1 - \cot^2 \theta$$

$$= 1 - \left(\frac{R}{X_0} \right)^2$$

$$\text{i.e. when } \omega_r = \omega_s \sqrt{1 - \left(\frac{R}{X_0} \right)^2}$$

but X_0 is the magnetizing reactance and typical values for $\frac{R}{X_0}$ are of the order of $\frac{1}{15}$.

$$\begin{aligned} \text{thus } \tau = 0 \text{ when } \omega_r &\approx \omega_s \sqrt{1 - \left(\frac{1}{15} \right)^2} \\ &= \omega_s \sqrt{.9956} \\ &= .998 \omega_s \end{aligned}$$

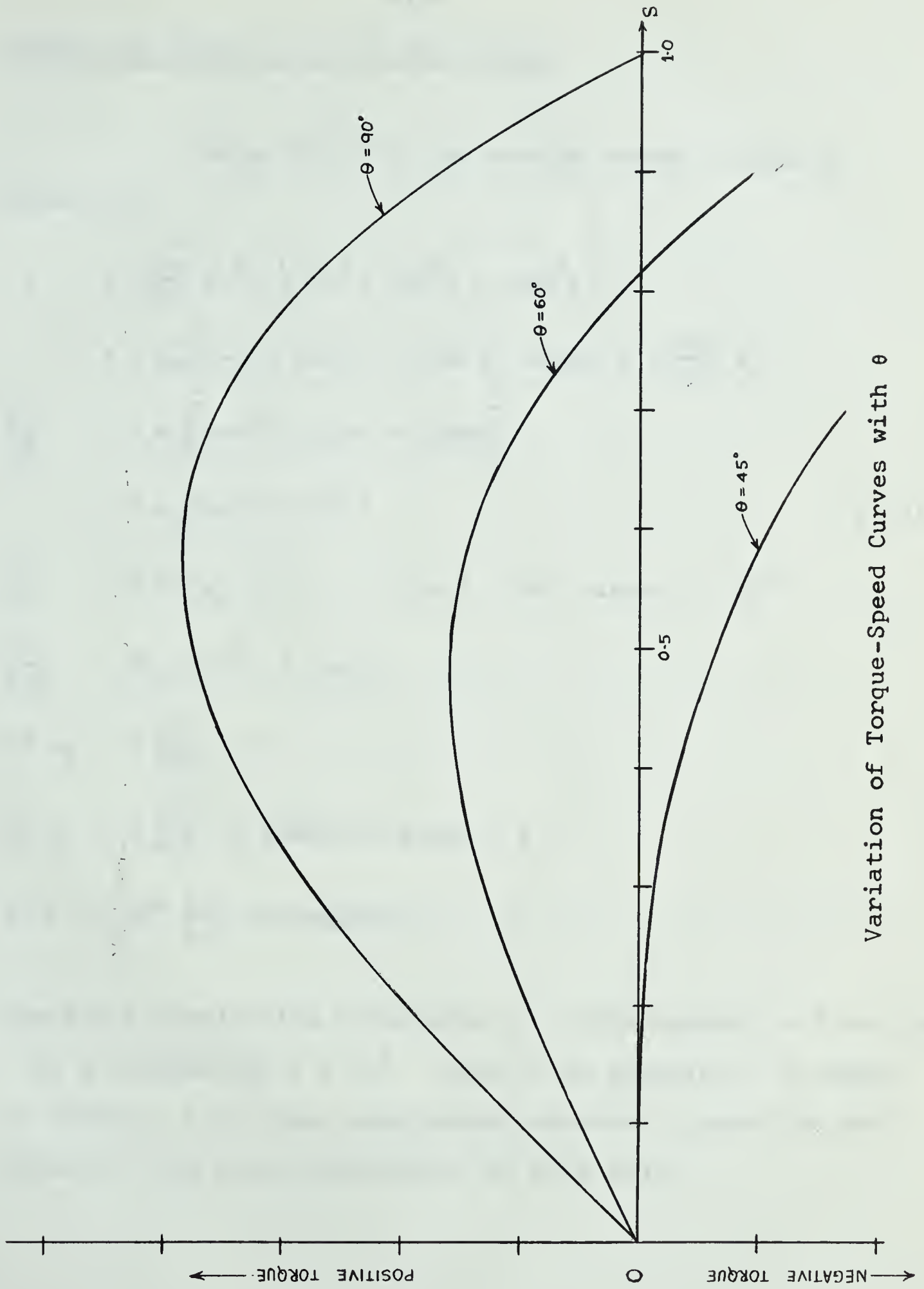
This proves that the machine develops zero torque at a speed which is slightly less than synchronous speed.

Torque-Speed Curves

Using Equation (37) sample calculations were done for the variation in τ with changes in θ . The data was obtained from C. G. Veinott⁽¹⁶⁾, page 365. A graph from those results is shown on the following page.

Zero Torque at Standstill

It was shown on page 30 that one condition for zero torque using the new theory was $\omega_r = 0$. This is in full agreement with practical results and proves that the single-phase induction motor has no starting torque.



Variation of Torque-Speed Curves with θ

Effect of Changes in R on the Torque

Using Eq.(37), the average steady torque is given by:

$$\begin{aligned}\tau &= \frac{k\pi r}{8\rho} \phi_t^2 \omega_r \left[(1-S^2) \sin^2 \theta - \cos^2 \theta \right] \\ &= K \omega_r \left[(1-S^2) \sin^2 \theta - \cos^2 \theta \right] \quad \text{where } K = \frac{k\pi r}{8\rho} \phi_t^2 \\ \frac{\partial \tau}{\partial \theta} &= K \omega_r \left[(1-S^2) \sin 2\theta + \sin 2\theta \right] \\ &= K \omega_r \sin 2\theta (2-S^2) \end{aligned} \quad \text{Eq.(42)}$$

$$\frac{\partial \tau}{\partial \theta} = 0 \text{ if } \omega_r = 0 \quad \theta = 0 \text{ or } \theta = 90^\circ \text{ since } \theta \leq 90^\circ$$

$$\frac{\partial^2 \tau}{\partial \theta^2} = K \omega_r (2-S^2) 2 \cos 2\theta$$

$$\text{if } \omega_r = 0 \quad \frac{\partial^2 \tau}{\partial \theta^2} = 0$$

$$\text{if } \theta = 0 \quad \frac{\partial^2 \tau}{\partial \theta^2} \text{ is positive since } S < 1$$

$$\text{if } \theta = 90^\circ \quad \frac{\partial^2 \tau}{\partial \theta^2} \text{ is negative}$$

Therefore, considering variations in τ with respect to θ we find τ is a maximum for $\theta = 90^\circ$. This is as expected. In order to obtain $\theta \approx 90^\circ$ the magnetizing reactance X_0 must be very large and the rotor resistance, R , very small.

Next, consider:

$$\begin{aligned}\frac{\partial \tau}{\partial \omega_r} &= K \left(\sin^2 \theta - 3 \left(\frac{\omega_r}{\omega_s} \right)^2 \sin^2 \theta - \cos^2 \theta \right) \\ &= K \left[\sin^2 \theta (1 - 3S^2) - 1 + \sin^2 \theta \right] \\ &= K \left[\sin^2 \theta (2 - 3S^2) - 1 \right]\end{aligned}$$

for

$$\frac{\partial \tau}{\partial \omega_r} = 0 \quad 2 \sin^2 \theta - 1 = 3S^2 \sin^2 \theta \quad \text{Eq. (43)}$$

$$\therefore S = \sqrt{\frac{2}{3} - \frac{1}{3 \sin^2 \theta}}$$

Thus, considering only variations in ω_r , τ is a maximum when:

$$\omega_r = \omega_s \sqrt{\frac{2}{3} - \frac{1}{3 \sin^2 \theta}} \quad \text{Eq. (44)}$$

The absolute value of τ maximum is given by the simultaneous equations:

$$\frac{\partial \tau}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial \tau}{\partial \omega_r} = 0$$

This means that τ maximum occurs when:

$$\begin{aligned}\omega_r &= \omega_s \sqrt{\frac{2}{3} - \frac{1}{3 \sin^2 90}} \\ &= \frac{\sqrt{3}}{3} \omega_s \\ &= .577 \omega_s\end{aligned}$$

$$\text{Eq. (45)}$$

As R increases $\sin^2 \theta$ decreases and the coefficient of ω_s in Eq.(44) decreases. This means that the speed at which maximum torque is developed is decreased. Also, from Eq.(37) we find that for any given speed τ decreases as R is increased. We can conclude that the effect of an increase in R in any given single-phase induction motor is to increase the slip at which the maximum torque is developed and to decrease the value of the maximum torque. This is in agreement with known practical results.

Comparison between steady-state torque expressions

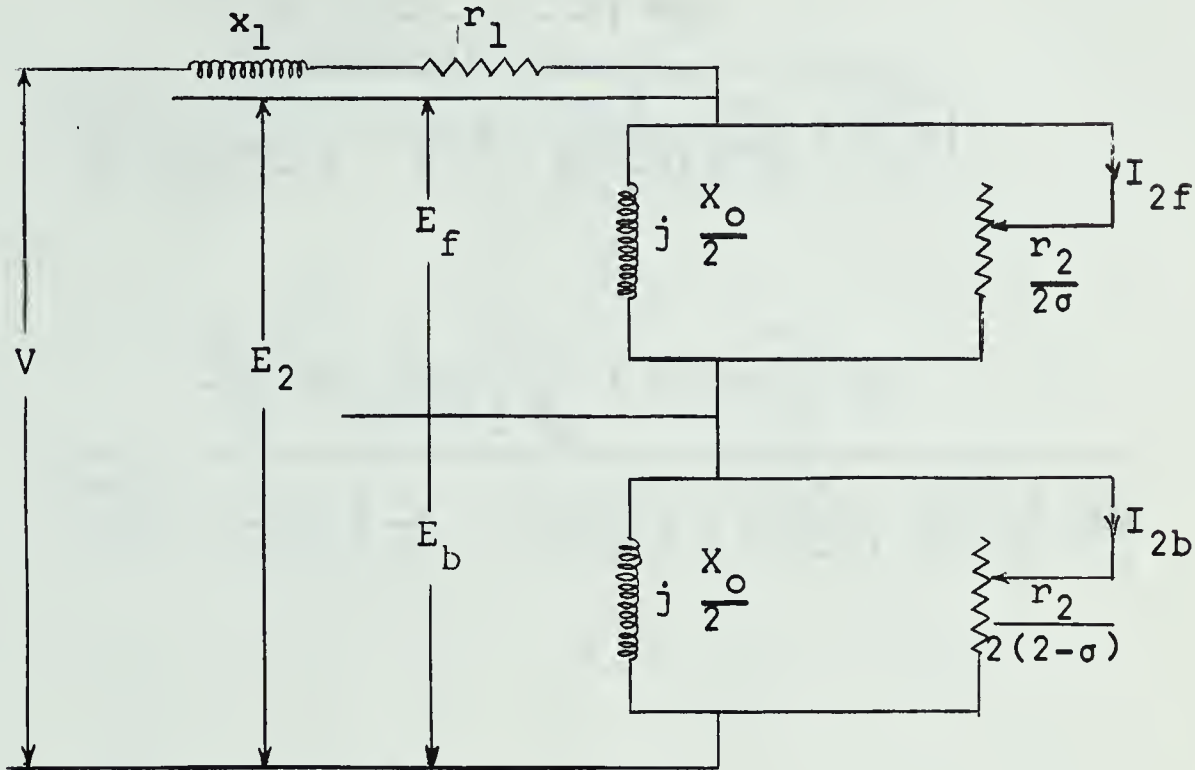


Figure 10

Equivalent Circuit for the Single-Phase Induction Motor,
according to the Revolving-Field Theory with
secondary leakage reactances neglected.

$$E_f = \frac{\frac{\frac{j r_2 X_0}{4\sigma}}{\frac{r_2}{2\sigma} + \frac{j X_0}{2}}}{\frac{\frac{j r_2 X_0}{4\sigma}}{\frac{r_2}{2\sigma} + \frac{j X_0}{2}} + \frac{\frac{j r_2 X_0}{4(2-\sigma)}}{\frac{r_2}{2(2-\sigma)} + \frac{j X_0}{2}}} E_2$$

$$\begin{aligned}
 E_f &= \frac{\frac{j r_2 X_o}{4\sigma} \left[\frac{r_2}{2(2-\sigma)} + \frac{j X_o}{2} \right] E_2}{\frac{j r_2 X_o}{4\sigma} \left[\frac{r_2}{2(2-\sigma)} + \frac{j X_o}{2} \right] + \frac{j r_2 X_o}{4(2-\sigma)} \left[\frac{r_2}{2\sigma} + \frac{j X_o}{2} \right]} \\
 &= \frac{\frac{j r_2 X_o}{4\sigma} \cdot \frac{X_o}{2(2-\sigma)} \left[\frac{r_2}{X_o} + j(2-\sigma) \right] E_2}{\frac{j r_2 X_o}{4\sigma} \cdot \frac{X_o}{2(2-\sigma)} \left[\frac{r_2}{X_o} + j(2-\sigma) \right] + \frac{j r_2 X_o}{4(2-\sigma)} \cdot \frac{X_o}{2\sigma} \left[\frac{r_2}{X_o} + j\sigma \right]} \\
 &= \frac{\left[\frac{r_2}{X_o} + j(2-\sigma) \right] E_2}{\left[\frac{r_2}{X_o} + j(2-\sigma) + \frac{r_2}{X_o} + j\sigma \right]} \\
 &= \frac{\left[\frac{r_2}{X_o} + j(2-\sigma) \right] E_2}{\frac{2r_2}{X_o} + j2} \\
 E_f^2 &= \frac{\left[\left(\frac{r_2}{X_o} \right)^2 + (2-\sigma)^2 \right] E_2^2}{4 \left[\left(\frac{r_2}{X_o} \right)^2 + 1 \right]}
 \end{aligned}$$

$$\begin{aligned}
 E_b &= \frac{\frac{\frac{j r_2 X_o}{4(2-\sigma)}}{\frac{r_2}{2(2-\sigma)} + \frac{j X_o}{2}}}{\frac{\frac{j r_2 X_o}{4(2-\sigma)}}{\frac{r_2}{2(2-\sigma)} + \frac{j X_o}{2}} + \frac{\frac{j r_2 X_o}{4\sigma}}{\frac{r_2}{2\sigma} + \frac{j X_o}{2}}} E_2 \\
 &= \frac{\frac{j r_2 X_o}{4(2-\sigma)} \left[\frac{r_2}{2\sigma} + \frac{j X_o}{2} \right] E_2}{\frac{j r_2 X_o}{4(2-\sigma)} \left[\frac{r_2}{2\sigma} + \frac{j X_o}{2} \right] + \frac{j r_2 X_o}{4\sigma} \left[\frac{r_2}{2(2-\sigma)} + \frac{j X_o}{2} \right]} \\
 &= \frac{\frac{j r_2 X_o}{4(2-\sigma)} \cdot \frac{X_o}{2\sigma} \left[\frac{r_2}{X_o} + j\sigma \right] E_2}{\frac{j r_2 X_o}{4(2-\sigma)} \cdot \frac{X_o}{2\sigma} \left[\frac{r_2}{X_o} + j\sigma \right] + \frac{j r_2 X_o}{4\sigma} \cdot \frac{X_o}{2(2-\sigma)} \left[\frac{r_2}{X_o} + j(2-\sigma) \right]} \\
 &= \frac{\left[\frac{r_2}{X_o} + j\sigma \right] E_2}{\left[\frac{r_2}{X_o} + j\sigma + \frac{r_2}{X_o} + j2-\sigma \right]}
 \end{aligned}$$

$$E_b = \frac{\left[\frac{r_2}{X_o} + j\sigma \right] E_2}{\frac{2r_2}{X_o} + j2}$$

$$E_b^2 = \frac{\left[\left(\frac{r_2}{X_o} \right)^2 + \sigma^2 \right] E_2^2}{4 \left[\left(\frac{r_2}{X_o} \right)^2 + 1 \right]}$$

$$\tau_f \propto \frac{E_f^2}{\frac{r_2}{2\sigma}}$$

$$\tau_f = \frac{k_1 \sigma E_f^2}{\frac{r_2}{2\sigma}}$$

$$= \frac{k_1 \sigma \left[\left(\frac{r_2}{X_o} \right)^2 + (2-\sigma)^2 \right] E_2^2}{2r_2 \left[\left(\frac{r_2}{X_o} \right)^2 + 1 \right]}$$

where k_1 is a constant

$$\tau_b \propto \frac{E_b^2}{\frac{r_2^2}{2(2-\sigma)}}$$

$$\tau_b = \frac{k_1 (2-\sigma) \left[\left(\frac{r_2}{X_o} \right)^2 + \sigma^2 \right] E_2^2}{2r_2 \left[\left(\frac{r_2}{X_o} \right)^2 + 1 \right]}$$

$$\tau = T_f - \tau_b$$

$$= \frac{k_1 E_2^2 \left[\sigma \left(\frac{r_2}{X_o} \right)^2 + \sigma (2-\sigma)^2 - (2-\sigma) \left(\frac{r_2}{X_o} \right)^2 - (2-\sigma) \sigma^2 \right]}{2r_2 \left[\left(\frac{r_2}{X_o} \right)^2 + 1 \right]}$$

$$= \frac{k_1 E_2^2 \left[2(\sigma-1) \left(\frac{r_2}{X_o} \right)^2 + 2(2-\sigma)(\sigma-\sigma^2) \right]}{2r_2 \left[\left(\frac{r_2}{X_o} \right)^2 + 1 \right]}$$

$$= \frac{k_1 E_2^2 (\sigma-1) \left[\left(\frac{r_2}{X_o} \right)^2 - 2\sigma + \sigma^2 \right]}{r_2 \left[\left(\frac{r_2}{X_o} \right)^2 + 1 \right]}$$

$$\tau = 0 \text{ if } \sigma = 1 \text{ or } \left[\sigma^2 - 2\sigma + \left(\frac{r_2}{X_0} \right)^2 \right] = 0$$

i.e. $\sigma = 1 \pm \sqrt{1 - \left(\frac{r_2}{X_0} \right)^2}$

Substituting $\left(1 - \frac{\omega_r}{\omega_s} \right)$ for σ and $\cot \theta$ for $\frac{r_2}{X_0}$

$$\text{we get } \tau = \frac{k_1 E_2^2 \sin^2 \theta}{r_2} \left(-\frac{\omega_r}{\omega_s} \right) \left[1 - \frac{2\omega_r}{\omega_s} + \left(\frac{\omega_r}{\omega_s} \right)^2 - 2 + \frac{2\omega_r}{\omega_s} + \cot^2 \theta \right]$$

$$= \frac{k_1 E_2^2}{r_2} \sin^2 \theta \frac{\omega_r}{\omega_s} \left[1 - \left(\frac{\omega_r}{\omega_s} \right)^2 - \cot^2 \theta \right]$$

$$= \frac{k_1 E_2^2 \omega_r}{r_2 s} \left[\left\{ 1 - \left(\frac{\omega_r}{\omega_s} \right)^2 \right\} \sin^2 \theta - \cos^2 \theta \right]$$

which is of the same form as that obtained from the new theory.

SUMMARY AND CONCLUSIONS

The theory presented was in agreement with the following known facts about the Single-Phase Induction Motor.

- 1) At standstill, both the pulsating torque and the useful torque are zero.
- 2) The machine develops zero torque at a speed which is slightly below synchronous speed.
- 3) The maximum torque developed varies inversely as the magnitude of the secondary resistance.
- 4) The machine develops negligible torque at very low speeds if enough external resistance is added to the rotor such that the total secondary resistance is approximately equal to the magnetizing reactance.

This investigation also shows that the average useful torque increases as θ is increased and, for any given speed, τ is a maximum when $\theta = 90^\circ$. Another factor which comes to light is that for θ less than 45° the machine cannot develop a positive torque. It was also observed that for any given speed the magnitude of the pulsating torque was a maximum for $\theta = 60^\circ$ and zero for $\theta = 90^\circ$.

Fortunately for the designer, the average useful torque increases and the average pulsating torque decreases as θ approaches 90° . For an increase in θ either X_0 is increased or R is decreased or both.

A. L. Kimball, Jr., and P.L. Alger⁽³⁾ did experiments to show that the pulsating torque was electromagnetic in its origin and it was present at no load. To this we can now add that the magnitude of this pulsating torque is a maximum for $\theta = 60^\circ$ and it decreases according to Eq.(41) reaching zero at $\theta = 90^\circ$.

REFERENCES

1. STEINMETZ, C.P.
Single-Phase Induction Motor. A.I.E.E. Transactions,
Vol. 15, 1898.
2. LAMME, B.G.
A physical conception of the operation of the Single-
Phase Induction Motor. *ibid*, Vol. 37, 1918.
3. KIMBALL, Jr., A.L. and ALGER, P.L.
Single-Phase Motor-Torque Pulsations. *ibid*. Vol. 43,
June 1924.
4. WEST, H.R.
The Cross-Field Theory of Alternating-Current
Machines. *ibid*. Vol. 45, 1926.
5. MORRILL, Wayne J.
The Revolving Field Theory of the Capacitor Motor.
ibid. Vol. 48, 1929.
6. BUTTON, C.T.
Single-Phase Motor Theory - A correlation of cross-
field and revolving-field concepts. *ibid*. 1940.
7. ALGER, P.L.
The Dilemma of Single-Phase Induction Motor Theory.
ibid. Vol. 77, 1958.
8. BRETCH, Edward
The Steinmetz conception of the Single-Phase Squirrel-
Cage Motor. *ibid*. Vol. 77, 1958.
9. KARAPETOFF, V.
On the equivalence of the two theories of the Single-
Phase Induction Motor. A.I.E.E. Journal, August, 1921.
10. PUCHSTEIN, A.F. and LLOYD, T.C.
Single-Phase Induction Motor Performance. *Electrical
Engineer*, Oct. 1937.
11. BEWLEY, L.V.
Alternating Current Machinery, Macmillan Co.
12. ALGER, P.L.
The Nature of Polyphase Induction Machines. John
Wiley & Sons.
13. PUCHSTEIN, LLOYD and CONRAD
Alternating-Current Machines. John Wiley & Sons
3rd ed.

14. MUELLER, G.V.
Alternating Current Machines. McGraw-Hill. 1st ed.
15. DAWES, C.L.
Electrical Engineering Vol. II. McGraw-Hill. 3rd ed.
16. VEINOTT, C.G.
Theory and Design of Small Induction Motors.
McGraw-Hill.
17. STEINMETZ, C.P.
Theoretical elements of Electrical Engineering.
McGraw-Hill. 3rd ed.
18. FITZGERALD, A.E. and KINGSLEY, Jr., C.
Electric Machinery. McGraw-Hill, 2nd ed.
19. LAWRENCE, R.R. and RICHARDS, H.E.
Principles of Alternating Current Machinery.
McGraw-Hill. 4th ed.
20. LANGSDORF, A.S.
Theory of Alternating Current Machinery. McGraw-Hill,
2nd ed.

APPENDIX I

$$I_1 \left[r_1 + j(x_1 + X_o) \right] - I_{2t}(jX_o) = E$$

$$I_1 (-jX_o) + I_{2s} \left[-S(x_2 + X_o) \right] + I_{2t} \left[r_2 + j(x_2 + X_o) \right] = 0$$

$$I_1 (-SX_o) + I_{2s} \left[r_2 + j(x_2 + X_o) \right] + I_{2t} \left[S(x_2 + X_o) \right] = 0$$

This is of the form

$$a I_1 + 0 + b I_{2t} = E$$

$$c I_1 + d I_{2s} + e I_{2t} = 0$$

$$\alpha I_1 + \beta I_{2s} + \gamma I_{2t} = 0$$

$$\begin{aligned} \therefore I_1 &= \frac{\begin{vmatrix} E & 0 & b \\ 0 & d & e \\ 0 & \beta & \gamma \end{vmatrix}}{\begin{vmatrix} a & 0 & b \\ c & d & e \\ \alpha & \beta & \gamma \end{vmatrix}} = \frac{E (d\gamma - \beta e)}{a (d\gamma - \beta e) + b(c\beta - \alpha d)} \\ &= \frac{A}{U_1 + jW_1} \text{ say} \end{aligned}$$

where,

$$\begin{aligned} A &= E \left[-S^2 (x_2 + X_o)^2 - \{r_2 + j(x_2 + X_o)\}^2 \right] \\ &= E \left[-S^2 (x_2 + X_o)^2 - r_2^2 - j2r_2(x_2 + X_o) + (x_2 + X_o)^2 \right] \end{aligned}$$

$$A = E \left[-r_2^2 + (1-S^2)(x_2+X_o)^2 - j2r_2(x_2+X_o) \right]$$

$$\therefore I_1 = \frac{E \left[-r_2^2 + (1-S^2)(x_2+X_o)^2 - j2r_2(x_2+X_o) \right]}{U_1 + jW_1}$$

$$I_{2t} = \frac{\begin{vmatrix} E & a & 0 \\ 0 & c & d \\ 0 & \alpha & \beta \end{vmatrix}}{\begin{vmatrix} a & 0 & b \\ c & d & e \\ \alpha & \beta & \gamma \end{vmatrix}} = \frac{E(c\beta - \alpha d)}{a(d\gamma - \beta e) + b(c\beta - \alpha d)}$$

$$= \frac{B}{U_1 + jW_1} \quad \text{say}$$

where,

$$B = E \left[\{r_2 + j(x_2+X_o)\} (-jX_o) - (-SX_o) \{-S(x_2+X_o)\} \right]$$

$$= E \left[-jr_2X_o + X_o(x_2+X_o) - S^2x_2X_o - S^2X_o^2 \right]$$

$$= EX_o \left[(1-S^2)(x_2+X_o) - jr_2 \right]$$

$$\therefore I_{2t} = \frac{EX_o \left[(1-S^2)(x_2+X_o) - jr_2 \right]}{U_1 + jW_1}$$

$$I_{2s} = \frac{\begin{vmatrix} a & E & b \\ c & 0 & e \\ \alpha & 0 & \gamma \end{vmatrix}}{\begin{vmatrix} a & 0 & b \\ c & d & e \\ \alpha & \beta & \gamma \end{vmatrix}} = \frac{-E (c\gamma - \alpha e)}{a(d\gamma - \beta e) + b(c\beta - \alpha d)}$$

$$I_{2s} = \frac{C}{U_1 + jW_1} \quad \text{say}$$

where,

$$\begin{aligned} C &= -E \left[(-jX_o) S(x_2 + X_o) - \{ r_2 + j(x_2 + X_o) \} (-SX_o) \right] \\ &= -E \left[(-jSx_2X_o - jSX_o^2 + r_2SX_o + jSX_o^2 + jSx_2X_o) \right] \\ &= -ESX_o r_2 \end{aligned}$$

$$\therefore I_{2s} = \frac{-ESX_o r_2}{U_1 + jW_1}$$

$$U_1 + jW_1 = a(d\gamma - \beta e) + b(c\beta - \alpha d)$$

$$\begin{aligned} &= \left[\{ r_1 + j(x_1 + X_o) \} \left(-S^2 (x_2 + X_o)^2 - \{ r_2 + j(x_2 + X_o) \}^2 \right) \right. \\ &\quad \left. + (-jX_o) \left((-jX_o) \{ r_2 + j(x_2 + X_o) \} - (-SX_o) \{ -S(x_2 + X_o) \} \right) \right] \end{aligned}$$

$$U_1 + jW_1 = \left[\left\{ r_1 + j(x_1 + X_0) \right\} \left\{ -s^2(x_2 + X_0)^2 - r_2^2 + (x_2 + X_0)^2 - j2r_2(x_2 + X_0) \right\} \right. \\ \left. - X_0^2 \left\{ r_2 + j(x_2 + X_0) \right\} - jSX_0^2 \left\{ -S(x_2 + X_0) \right\} \right] \\ = \left[-r_1 s^2(x_2 + X_0)^2 - r_1 r_2^2 + r_1(x_2 + X_0)^2 + 2r_2(x_1 + X_0)(x_2 + X_0) - r_2 X_0^2 \right. \\ \left. + j \left\{ -s^2(x_1 + X_0)(x_2 + X_0)^2 - r_2^2(x_1 + X_0) + (x_1 + X_0)(x_2 + X_0)^2 \right\} \right. \\ \left. + j \left\{ -X_0^2(x_2 + X_0) + S^2 X_0^2(x_2 + X_0) - 2r_1 r_2(x_2 + X_0) \right\} \right]$$

$$= \left[-r_1 r_2^2 + r_1(x_2 + X_0)^2 - r_1 s^2(x_2 + X_0)^2 \right. \\ \left. + 2r_2 x_1(x_2 + X_0) + 2r_2 X_0(x_2 + X_0) - r_2 X_0^2 \right. \\ \left. + j \left\{ -r_2^2(x_1 + X_0) - 2r_1 r_2(x_2 + X_0) + x_1(x_2 + X_0)^2 + X_0(x_2 + X_0)^2 \right\} \right. \\ \left. + j \left\{ +(S^2 - 1) X_0^2(x_2 + X_0) - x_1 s^2(x_2 + X_0)^2 - X_0 s^2(x_2 + X_0)^2 \right\} \right]$$

$$\therefore U_1 = \left[-r_1 r_2^2 + (1 - S^2) r_1(x_2 + X_0)^2 + 2r_2 x_1(x_2 + X_0) \right. \\ \left. + 2r_2 x_2 X_0 + 2r_2 X_0^2 - r_2 X_0^2 \right]$$

$$= \left[-r_1 r_2^2 + 2r_2 x_1(x_2 + X_0) + r_2 X_0(X_0 + 2x_2) + (1 - S^2) r_1(x_2 + X_0)^2 \right]$$

$$W_1 = \left[-r_2^2(x_1 + X_0) - 2r_1 r_2(x_2 + X_0) + (1 - S^2) x_1(x_2 + X_0)^2 \right. \\ \left. + (1 - S^2)(x_2 X_0 + X_0^2)(x_2 + X_0) + (S^2 - 1) X_0^2(x_2 + X_0) \right]$$

$$= \left[-r_2^2(x_1 + X_0) - 2r_1 r_2(x_2 + X_0) \right. \\ \left. + (1 - S^2) \{ x_1(x_2 + X_0)^2 + x_2 X_0(x_2 + X_0) \} \right]$$

APPENDIX 2

Double Frequency Secondary Voltage

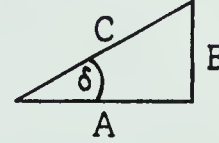
An expression for the r.m.s.value of the secondary voltage is derived below:

$$\sin(\omega_s t - \theta) = \sin \omega_s t \cos \theta - \cos \omega_s t \sin \theta$$

$$\cos(\omega_s t - \theta) = \cos \omega_s t \cos \theta + \sin \omega_s t \sin \theta$$

$$\begin{aligned} e_{i\alpha} &= k \left[-\omega_r \phi_t \sin \omega_s t \sin \alpha + \omega_s \phi_t \cos \omega_s t \cos \alpha \right. \\ &\quad \left. + \omega_r \phi_s \sin(\omega_s t - \theta) \cos \alpha + \omega_s \phi_s \cos(\omega_s t - \theta) \sin \alpha \right] \\ &= k \left[\left\{ -\omega_r \phi_t \sin \alpha + \omega_r \phi_s \cos \alpha \cos \theta + \omega_s \phi_s \sin \alpha \sin \theta \right\} \sin \omega_s t \right. \\ &\quad \left. + \left\{ \omega_s \phi_t \cos \alpha - \omega_r \phi_s \cos \alpha \sin \theta + \omega_s \phi_s \sin \alpha \cos \theta \right\} \cos \omega_s t \right] \end{aligned}$$

$$= A \sin \omega_s t + B \cos \omega_s t$$



where,

$$A = k \left[-\omega_r \phi_t \sin \alpha + \omega_r \phi_s \cos \alpha \cos \theta + \omega_s \phi_s \sin \alpha \sin \theta \right]$$

and

$$B = k \left[\omega_s \phi_t \cos \alpha - \omega_r \phi_s \cos \alpha \sin \theta + \omega_s \phi_s \sin \alpha \cos \theta \right]$$

Let,

$$C = \sqrt{A^2 + B^2}$$

Then,

$$e_{i\alpha} = C \cos \delta \sin \omega_s t + C \sin \delta \cos \omega_s t$$

$$e_{i\alpha} = C \sin (\omega_s t + \delta)$$

Let,

$$\begin{aligned} e_{1r.m.s.} &= \sqrt{\frac{2}{T'} \int_0^{T'} C^2 \sin^2 (\omega_s t + \delta) dt} \\ &= \frac{C^2}{T'} \int_0^{T'} \left\{ 1 - \cos 2 (\omega_s t + \delta) \right\} dt \\ &= \frac{C^2}{T'} \left[\frac{T'}{2} - \int_0^{T'} \cos 2 (\omega_s t + \delta) dt \right] \\ &= \frac{C^2}{2} - \frac{C^2}{T'} \int_0^{\frac{\omega_s T'}{2} + \delta} \cos 2z \frac{dz}{\omega_s} \text{ where } z = (\omega_s t + \delta) \\ &= \frac{C^2}{2} - \frac{C^2}{2\omega_s T'} \left[\sin 2z \right]_{\delta}^{\frac{\omega_s T'}{2} + \delta} \\ &= \frac{C^2}{2} + \frac{C^2}{2\omega_s T'} \sin 2\delta - \frac{C^2}{2\omega_s T'} \sin (\omega_s T' + 2\delta) \\ &= \frac{C^2}{2} + \frac{C^2}{\pi} \sin 2\delta \end{aligned}$$

$$e_{2r.m.s.} = \sqrt{\frac{2}{T'} \int_{\frac{T'}{2}}^{T'} C^2 \sin^2 (\omega_s t + \delta) dt}$$

Similarly, it can be shown that

$$e_{2r.m.s.} = \frac{C^2}{2} - \frac{C^2}{2\omega_s T'} \left[\sin 2z \right]_{\frac{\omega_s T'}{2} + \delta}^{\omega_s T' + \delta}$$

$$\begin{aligned}
 e_{2\text{r.m.s.}} &= \frac{C^2}{2} - \frac{C^2}{2\omega_s T'} \left[\sin(2\omega_s T' + 2\delta) - \sin(\omega_s T' + 2\delta) \right] \\
 &= \frac{C^2}{2} - \frac{C^2}{2\omega_s T'} \left[\sin 2\delta + \sin 2\delta \right] \\
 &= \frac{C^2}{2} - \frac{C^2}{\pi} \sin 2\delta
 \end{aligned}$$

This shows that $e_{2\text{r.m.s.}}$ is made up of a value $\frac{C^2}{2}$ and a double frequency pulsating value of $\frac{C^2}{\pi} \sin 2\delta$.

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